EXTENDING MATCHINGS IN GRAPHS: A SURVEY

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ABSTRACT

Gallai and Edmonds independently obtained a canonical decomposition of graphs in terms of their maximum matchings. Unfortunately, one of the degenerate cases for their theory occurs when the graph in question has a *perfect* matching (also known as a 1-*factor*). Kotzig, Lovász and others subsequently developed a further decomposition of such graphs. Among the "atoms" of this decomposition is the family of *bicritical* graphs. (A graph G is **bicritical** if $G - u - v$ has a perfect matching for every choice of two points u, v in G.) So far such graphs have resisted further decomposition procedures.

Motivated by these mysterious graphs, we introduced the following definition. Let p and n be positive integers with $n \leq (p-2)/2$. Graph G is n-**extendable** if G has a matching of size n and every such matching *extends* to (i.e., is a subset of) a perfect matching in G. It is clear that if a graph is bicritical it is 1-extendable. A more interesting result is that if a graph is 2-extendable, it is either bipartite or bicritical. It is also true that if a graph is n-extendable, it is also $(n-1)$-extendable. Hence for non-bipartite graphs we have a nested sequence of families of bicritical graphs to study.

In this paper, we will survey a variety of results obtained over the past few years concerning n-extendable graphs. In particular, we will describe how the property of n-extendability interacts with such other graph parameters as genus, toughness, claw-freedom and degree sums and generalized neighborhood conditions. We will also investigate the behavior of matching extendability under the operation of Cartesian product. The study of n-extendability for *planar* graphs has been—and continues to be—of particular interest.

1. Introduction, Terminology and some Motivation

For any terminology not defined in this paper, the reader is directed to either [23] or [3]. Let G be any connected graph. A set of lines $M \subseteq E(G)$ is called a **matching** if they are independent; i.e., no two of them share a common endpoint. A matching is said to be **perfect** if it covers all points of G. (Hereafter in this paper, "perfect matching" will often be abbreviated as "**pm**".) Let $\Phi(G)$ denote the number of perfect matchings in graph G. There are other areas of science (see [23]) in which it is of interest to determine $\Phi(G)$. But it seems very unlikely that an efficient algorithm for computing $\Phi(G)$ will ever be found. In particular, Valiant [48] proved the following result concerning the complexity of this task.

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Theorem 1.1. The problem of determining $\Phi(G)$ is #P-complete–and hence NP-hard–even when G is bipartite. ∎

So, as is so often the case in mathematics, when a function cannot be computed exactly, we turn instead to a search for bounds. In particular, the material to follow in this paper can be said to be motivated by the search for a non-trivial *lower* bound for the parameter $\Phi(G)$.

It was noticed first by Lovász [20] that a class of graphs called *bicritical* play an important role in bounding the number of perfect matchings. (A graph G is said to be **bicritical** if $G - u - v$ has a perfect matching for every choice of a pair of points, u and v.)

Theorem 1.2. If G is k-connected and contains a perfect matching, but is not bicritical, then $\Phi(G) \geq k!$. ∎

In the author's opinion, the role of the property of bicriticality in the above theorem is somewhat counterintuitive, and therefore intriguing. After all, it is trivial to see that a bicritical graph has the property that each of its lines lies in a perfect matching. So why should not bicritical graphs have an "enormous" number of perfect matchings, when in a sense, the opposite is true? It is no surprise, then, to note that bicritical graphs have played (and continue to play) an important role in studies involving a lower bound for $\Phi(G)$.

In the past twenty years or so, considerable effort has been devoted to developing a *canonical decomposition theory* for graphs with perfect matchings. We shall now present the barest of outlines of this effort and refer the interested reader to [23]–and to the list of references to be found therein–for a much more detailed treatment.

Let G be a graph containing a perfect matching and suppose n is a positive integer such that $n < |V(G)|/2$. Graph G will be called **n-extendable** if every set of n independent lines extends to (i.e., is a subset of) a pm.

Let us begin our discussion of the decomposition theory by assuming that each line of the graph G under consideration lies in at least one pm for G. (In other words, assume that G is 1-extendable. Such graphs are also sometimes called **matching-covered**. See Lovász [21].) Clearly, lines lying in no pm can be ignored when attempting to count the total number of pm's in the graph. Moreover, the determination of all such "forbidden" lines can be carried out in polynomial time by applying Edmonds matching algorithm at most $|E(G)|$ times.

Now let us encode each perfect matching in G as a binary vector of length $|E(G)|$, where the jth entry is a 1 if line j of G belongs to the pm and is a 0, if it does not. Take the linear span of all such binary vectors over the reals, $\Re$. The dimension of this space is called the **real rank of** G and is denoted by $r_{\Re}(G)$. Clearly, $r_{\Re}(G) \leq \Phi(G)$ and hence we have a lower bound of the type sought.

But can the quantity $r_{\Re}(G)$ be efficiently computed? The answer to this question is "yes", fortunately, but we need to lay a bit more groundwork first. Let us call a bicritical graph which, in addition, is 3-connected, a **brick**. It is a fact–although a highly non-trivial one–that a decomposition theory exists for graphs with perfect matchings which

terminates in a list of bricks associated with the parent graph. Moreover, this list of bricks is an invariant of the graph and the list can be determined in *polynomial* time.. (See [22].) Denote the *number* of such bricks of G by $\beta(G)$. We then can exactly compute $r_{\mathfrak{R}}(G)$ via the following beautiful result due to Edmonds, Lovász and Pulleyblank [5].

Theorem 1.3. If G is any 1-extendable graph, then

$$r_{\mathfrak{R}}(G) = |E(G)| - |V(G)| + 2 - \beta(G).$$

∎

The careful reader may have noticed by now that it follows immediately from the definition of a bicritical graph that no bipartite graph can be bicritical, let alone a brick. On the other hand, it is easy to find 1-extendable *bipartite* graphs. Such graphs will yield no bricks in the decomposition procedure, but it is worthwhile to note that the equation of Theorem 1.3 still holds, although in this case, $\beta(G) = 0$. This result, for the special case of bipartite graphs, actually predates the general formula of Theorem 1.3 and is due to Naddef [26].

Corollary 1.4. If G is any 1-extendable bipartite graph, then

$$r_{\mathfrak{R}}(G) = |E(G)| - |V(G)| + 2.$$

∎

In the most recent version of the decomposition procedure referred to above (and called the *tight cut decomposition procedure* by Lovász [22]), in addition to the invariant list of bricks obtained, there is a second list of building blocks called *braces*. Although these graphs do not figure in the rank formula stated above, as do the bricks, they are deserving of mention in a paper on matching extension. In particular, a bipartite graph is a **brace** if it is 2-extendable.

At this point, let us stop to ask the question: are there any well-known classes of graphs for which $\Phi(G)$ can always be *exactly* determined in *polynomial* time? The best known such class is the class of planar graphs. This was proved long before the development of the decomposition theory discussed above by Kasteleyn [14, 15] who also gave an algorithm for counting the pm's of a planar graph. Although this procedure was presented before complexity of algorithms attracted much attention, fortunately Kasteleyn's algorithm is easily seen to be polynomial. Kasteleyn showed that if one could direct the lines of an undirected graph G so as to obtain a *Pfaffian orientation* of G, Then $\Phi(G)$ was just the value of the determinant of a certain matrix associated with the oriented graph, and hence obtainable in polynomial time. He then showed that one could always find such an orientation when graph G was planar. (For the definition of a Pfaffian orientation, as well as a more detailed discussion of the so-called "Kasteleyn method", see [23; Chapter 8].)

In a much more recent result, Vazirani and Yannakakis [49] have demonstrated the following important relationship between the bricks and braces of a graph and Pfaffian orientations.

Theorem 1.5. An arbitrary graph G has a Pfaffian orientation if and only if all of its bricks and braces have such an orientation. ∎

There are several interesting and closely related complexity questions about Pfaffian orientations of graphs which remain unresolved. (1) Does a given graph have a Pfaffian orientation? (2) Is a given orientation of a graph a Pfaffian orientation? Recently, Vazirani and Yannakakis [49] have demonstrated that (1) and (2) are polynomially equivalent. In fact, in the case of bipartite graphs, Questions (1) and (2) are polynomially equivalent to yet a third unsettled question: (3) Given a directed graph, does it contain a directed cycle of even length?

The tight set decomposition procedure yielding the canonical lists of bricks and braces is, to be sure, a deep and beautiful theory. But note that when the graph with which one starts is itself a brick or a brace, the theory provides no further "decomposition". Indeed, at this point we come up against something of a "brick wall"! (Of course the pun is intentional!) At present, no theory for further decomposing bricks and braces exists. On the other hand, the following result is known. (See [33].)

Theorem 1.6. If G is 2-extendable then G is either a brick or a brace. ∎

So what can we say about the structure of 2-extendable graphs?

Before we cease posing such questions and start pursuing answers, let us note the next two results, the proofs of which can also be found in [33].

Theorem 1.7. If $n \geq 1$ and G is n-extendable, then G is $(n+1)$-connected. ∎

Theorem 1.8. If $n \geq 2$ and G is n-extendable, then G is $(n-1)$-extendable. ∎

For each $n \geq 1$, let us denote by $\mathcal{E}_n$ the class of all n-extendable graphs. Then Theorem 1.8 implies that these classes are "nested" as follows:

$$\mathcal{E}_1 \supset \mathcal{E}_2 \supset \mathcal{E}_3 \supset \cdots.$$

Moreover, if we let $\mathcal{B}$ denote the class of all bicritical graphs, then Theorems 1.6 and 1.8 imply that if G is *not* bipartite, the class of bicritical graphs can be included in the nesting:

$$\mathcal{E}_1 \supset \mathcal{B} \supset \mathcal{E}_2 \supset \mathcal{E}_3 \cdots.$$

(It is easily seen that all subset containments indicated above are proper.)

This leads to our final motivational question. What can one say about the structure of n-extendable graphs?

With this question in mind, we now proceed to survey a number of results relating the concept of n-extendability to other well-known graph parameters.

2. Extendability and Genus

Our interest in matching extendability versus surface embedding began with the following result [35].

Theorem 2.1. No planar graph is 3-extendable. ∎

To proceed to the study of extendability on surfaces of higher genus, some notation and a definition are needed.

Let Σ denote a surface, either orientable or non-orientable. Then let $\mu(\Sigma)$ denote the smallest integer such that no graph G embeddable in surface Σ is $\mu(\Sigma)$-extendable. For example, the dodecahedron is easily seen to be 2-extendable, while by Theorem 2.1 above, no planar graph is 3-extendable, so it follows that $\mu(\text{sphere}) = 3$.

Recall that the **Euler Characteristic** of surface Σ is defined by $\chi(\Sigma) = 2 - 2\gamma$, when Σ is orientable and $2 - \gamma$, when Σ is not orientable. (Here γ denotes the genus of the surface.)

The next two results generalize Theorem 2.1 to surfaces of genus > 0.

Theorem 2.2. [36] (a) If Σ is an orientable surface with genus $\gamma > 0$, then

$$\mu(\Sigma) \leq \frac{9}{2} + \left\lfloor \frac{18(\gamma - 1)}{7 + \sqrt{48\gamma - 47}} \right\rfloor,$$

and if (b) in addition, graph G is triangle-free, G is not $(2 + \lfloor 2\sqrt{\gamma} \rfloor)$-extendable. ∎

In a result yet to appear at the time of this writing, Dean [4] has extended the above result in several ways.

Theorem 2.3. (a) If Σ is an orientable surface of genus $\gamma > 0$, then

$$\mu(\Sigma) = 2 + \lfloor 2\sqrt{\gamma} \rfloor,$$

while (b) if Σ is non-orientable of genus $\gamma > 0$, then

$$\mu(\Sigma) = 2 + \lfloor \sqrt{2\gamma} \rfloor.$$

∎

It should be noticed that Dean's results are sharp; that is, he has proved equality in both cases (a) and (b). In particular, this implies that Dean's upper bound on n-extendability of a graph with genus $\gamma > 0$ is independent of whether or not G is triangle-free.

Although Theorem 2.3 significantly extends and improves Theorem 2.2, the proof techniques used for both are very similar. In particular, both proofs make heavy use of what has come to be called *the theory of Euler contributions*. (For that matter, so did the proof of Theorem 2.1.) Since this approach has proved useful to the author, not only in the area of matching extension, but elsewhere as well [40], we shall sketch the main ideas. Perhaps some of readers of this paper will find new ways to exploit it in their own work.

The theory of Euler contributions was first studied by Lebesgue [17] and later by Ore [29], as well as by Ore and the author [30]. A well-known result due to Youngs [50] states that *every* embedding of a graph G in its surface of minimum *orientable* genus is 2-cell. It is much less widely known, and to the best of the author's knowledge, was proved for the first time only in 1987 by Parsons, Pica, Pisanski and Ventre [31], that *at least*

one embedding of a graph G in its surface of minimum *non-orientable* genus is 2-cell. For our purposes, the important fact to be gleaned from this discussion about minimum embeddings is that when a graph is so embedded (orientably or non-orientably), Euler's formula (i.e., $p - q + r = \chi(\Sigma)$) holds. (See Massey [24].)

Let v be any point in graph G. The **Euler contribution of point** v, $\Phi(v)$ is defined by:

$$\Phi(v) = 1 - \frac{\deg_G v}{2} + \sum_{i=1}^{\deg v} \frac{1}{x_i},$$

where x_i denotes the "size" of the ith face at point v. (that is, the number of lines in the cycle which forms the boundary of the ith face).

The following lemma is clear.

Lemma 2.4. If graph G is minimally embedded surface Σ, then

$$\sum_{v \in V(G)} \Phi(v) = \chi(\Sigma).$$

∎

So it follows immediately that for some point $v \in V(G)$, we have $\Phi(v) \geq \chi(\Sigma)/|V(G)|$. We call such a point a **control point**.

It may be shown that there are limitations to the types of face configurations which may surround a control point. (Again, for details we refer the reader to [30].) The idea then is to find a partial matching which covers all the neighbors of point v–but not v itself–and is of minimum size subject to this covering demand. Obviously, then, such a partial matching cannot extend to a pm, for there is no way to cover the control point v with such an extension.

3. Extendability and Claw-free Graphs

Our next results are of a "forbidden" subgraph nature. In particular, we will consider so-called *claw-free* graphs. A graph G is said to be **claw-free** if it contains no induced subgraph isomorphic to the complete bipartite graph $K_{1,3}$–the so-called "claw". Perhaps the first deep theorem concerning claw-free graphs was due independently to Minty [25] and Sbihi [42] who showed that in any claw-free graph the independence number (also known as the stability number and the vertex-packing number) can be computed in polynomial time. Since the appearance of this result, studies involving claw-free graphs have appeared in abundance.

Part (a) of the following theorem was proved independently by Sumner [44] and Las Vergnas [16]; parts (b) and (c) are due to the author [38] .

Theorem 3.1. Suppose $n \geq 0$ and G is a $(2n+1)$-connected claw-free graph with $|V(G)|$ even. Then:

(a) If $n = 0$, then graph G has a perfect matching;
(b) If $n = 1$, then graph G is a brick (and hence also 1-extendable);
(c) If $n \geq 2$, then graph G is n-extendable. ∎

This theorem is sharp in the sense that, for all $n \geq 1$, we can construct a claw-free graph which is $2n$-connected, has an even number of points, but is *not* n-extendable.

Now what about some kind of "converse" to the above theorem? To this end, we have the following result. Let $\delta(G)$ denote the minimum degree in G.

Theorem 3.2. Suppose $n \geq 1$ and that G is a claw-free n-extendable graph with $|V(G)|$ even. Then $\delta(G) \geq 2n$. ∎

The lower bound on $\delta(G)$ in Theorem 3.2 is sharp. That is, for each $n \geq 1$, we can construct a graph H'_n which is n-extendable, claw-free and has $\delta(H'_n) = 2n$.

For proofs, constructions and further discussion about extending matchings in claw-free graphs, see [38].

4. Extendability in Products of Graphs

It was noticed sometime in the mists of the past by the author that the *3-cube* Q_3 is nice example of a small graph which is 2-extendable and bipartite, and hence a *brace*. In collaboration with Gyõri [9], we have been able to obtain a rather substantial generalization of this observation. Let G_1 and G_2 be any two graphs. The **Cartesian product** of G_1 and G_2 is denoted by $G_1 \times G_2$ and defined as follows. The point set $V(G_1 \times G_2) = \{(x_1, x_2) | x_1 \in V(G_1), x_2 \in V(G_2)\}$ and two points of the product (x_1, x_2) and (y_1, y_2) are adjacent in the product graph if either $x_1 = y_1$ and x_2y_2 is a line in G_2 or $x_2 = y_2$ and x_1y_1 is a line in G_1.

Theorem 4.1 Suppose k_1 and k_2 are two non-negative integers and that the two graphs G_i are k_i-extendable, for $i = 1, 2$. Then $G_1 \times G_2$ is $(k_1 + k_2 + 1)$-extendable. ∎

This result is sharp in the following sense. Suppose G_i is k_i-extendable for $i = 1, 2$ and suppose $\deg_{G_1} v = k_1 + 1$ and $\deg_{G_2} v = k_2 + 1$. Then $\deg_{G_1 \times G_2} v = k_1 + k_2 + 2$ and hence $\kappa(G_1 \times G_2) \leq k_1 + k_2 + 2$. On the other hand, the product graph $G_1 \times G_2$ is $(k_1 + k_2 + 1)$-extendable by the above theorem, and hence by Theorem 1.7, we have $\kappa(G_1 \times G_2) \geq k_1 + k_2 + 2$. Putting these two inequalities together, we have $\kappa(G_1 \times G_2) = k_1 + k_2 + 2$ and hence again by Theorem 1.7, graph $G_1 \times G_2$ is *not* $(k_1 + k_2 + 2)$-extendable.

5. Degree Sums and Neighborhood Unions

It seems the first so-called *degree sum* theorems are due to Ore [27, 28]. We combine three of his results in the next theorem.

Theorem 5.1. If G is a graph with p points such that for each pair of non-adjacent points u and v, $\deg u + \deg v \geq p$ (respectively, $\geq p - 1$, $\geq p + 1$), then G has a Hamiltonian cycle (respectively, has a Hamilton path, is Hamiltonian connected). ∎

During recent years, there has been a flurry of activity in the area of degree sum studies. (For a sampler of these, see [18].) Note that Ore's results involve degree sums of sets of two independent points. One of the directions of generalization which has occurred involves considering the degree sums of sets of t independent points, for $t \geq 3$. The next result is a theorem of this type. (See [39] for the proof.)

Theorem 5.2. Suppose G is a k-connected graph with p points where p is even and n is any integer satisfying $1 \leq n < p/2$. Suppose further that there exists a $t, 1 \leq t \leq k - 2n + 2$ such that for all independent sets $I = \{w_1, \ldots, w_t\}$ having $|I| = t$, it follows that $\sum_{i=1}^{t} \deg w_i \geq t((p-2)/2 + n) + 1$.

Then if

(a) $n = 1$, G is bicritical (and hence 1-extendable) and if

(b) $n \geq 2$, G is n-extendable. ∎

Theorem 5.2 is sharp in the following sense. Choose $n \geq 1$ and $k \geq 2n$. Define a graph $H(n,k)$ consisting of a complete graph on k points each of which is also adjacent to each member of an independent set $\{w_1, \ldots, w_{k-2n+1}, w_{k-2n+2}\}$ of cardinality $k - 2n + 2$. Let $t = k - 2n + 1$ and let $I = \{w_1, \ldots, w_t\}$. Then $\sum_{i=1}^{t} w_i = t((p-2)/2 + n)$, But graph $H(n,k)$ is *not* n-extendable.

It is perhaps instructive to present the following corollary of Theorem 5.2 (namely, the case when $t = 2$) in order to exhibit a result which more closely resembles those of Ore stated above.

Corollary 5.3. Let G be a graph with p points, p even, and let n be an integer, $1 \leq n < p/2$. Suppose that for all pairs of non-adjacent points u and v in G, $\deg u + \deg v \geq p + 2n - 1$. Then if

(a) $n = 1$, G is bicritical (and hence 1-extendable), and if

(b) $n \geq 2$, G is n-extendable. ∎

Let us mention one more special case. Bondy [2] has proved the following result which involves degree sums of sets of *three* independent points.

Theorem 5.4. If G is a 2-connected graph with p points such that for every set of three independent points w_1, w_2 and w_3 in G, $\deg w_1 + \deg w_2 + \deg w_3 \geq 3p/2$, then G has a Hamilton cycle. ∎

Let us compare this to the following corollary of Theorem 5.2.

Corollary 5.5. If G has p points, p even, is 3-connected and if for all independent triples of points w_1, w_2 and w_3, $\deg w_1 + \deg w_2 + \deg w_3 \geq 3p/2 + 1$, then G is bicritical. ∎

It is of some interest to ask if one really needs the assumption that G is 3-connected in Corollary 5.5. Bondy needed only to assume that the graphs in question be 2-connected. Also note that a bicritical graph is necessarily 2-connected.

The answer is "yes", we *do* need 3-connectivity. At least if $p \geq 10$, we can exhibit a (precisely) 2-connected graph which satisfies the degree sum bound for every triple of independent points, but which is not bicritical. (For details, see [39].)

Now let us turn our attention to the concept of *neighborhood union*. The following result is a neighborhood union analogue of Theorem 5.2 above.

Theorem 5.6. Let G be a k-connected graph on p points, p even, and $1 \leq n < p/2$. Suppose there is a t, $1 \leq t \leq k-2n+2$, such that for all independent sets $I = \{w_1, \ldots, w_t\}$ with $|I| = t$, it follows that $|\cup_{i=1}^{t} N(w_i)| \geq p - k + 2n - 1$. Then
(a) if $n = 1$, G is bicritical (and hence 1-extendable), and
(b) if $n \geq 2$, G is n-extendable. ∎

Let us now consider a neighborhood union result recently obtained by Faudree, Gould, Jacobson and Schelp [7].

Theorem 5.7. If G is a 2-connected (respectively, 3-connected) graph with p points, $p \geq 3$, and if for all pairs of non-adjacent points u and v, $|N(u) \cup N(v)| \geq (2p-1)/3$ (respectively $2p/3$), then G is Hamiltonian (respectively Hamiltonian-connected). ∎

Note that the following corollary follows immediately from Theorem 5.7.

Corollary 5.8. If G is a 3-connected graph on p points and if for all pairs of non-adjacent points u and v, $|N(u) \cup N(v)| \geq 2p/3$, then G is bicritical. ∎

Note that if $k \leq p/3 + 1$ and $n = 1$, Corollary 5.8 "beats" the result of our Theorem 5.6, part (a) when $t = 2$.

This observation suggests the following question. Suppose $n \geq 2$ is fixed. Is there a constant c, $0 < c < 1$, such that if $|N(u) \cup N(v)| \geq cp$, for all pairs of non-adjacent points u and v, then G is n-extendable? The answer is "no" and again the reader is referred to [39] for details.

6. Extendability and Toughness

Let S be a point cutset in a graph G and let $c(G-S)$ denote the number of components of $G-S$. Then, if G is not complete, the **toughness** of G is defined to be min $|S|/c(G-S)$ where the minimum is taken over all such point cutsets S of G. (The toughness of any complete graph is defined to be $+\infty$.) We denote the toughness of graph G by tough (G).

Theorem 6.1. Suppose G has p points, p even, and n is an integer such that $1 \leq n < p/2$. Then if tough $(G) > n$, graph G is n-extendable. ∎

This lower bound on toughness in the above theorem is sharp for all n. In fact, an infinite family of extremal graphs is very simple to describe in this case. For each n, join each of the $2n$ points of the complete graph K_{2n} to each point of two disjoint copies of the complete graph K_{2n+1}. The resulting graph has $6n+2$ points and easily seen to have

toughness $= n$. However, it is not n-extendable. (In fact, *no* set of n lines in the graph extends to a pm.)

If one now seeks some kind of converse to Theorem 6.1, one soon discovers that there is no lower bound on the toughness of the class of n-extendable graphs. A construction illustrating this fact is described in [34]. However, the number of points in a typical member of this extremal class is quite large, so one might amend the question to ask if $p = |V(G)|$, is there a "reasonable" function $f(p)$ such that if graph G is $f(p)$-extendable, then G has, say, toughness ≥ 1. In this case, one can answer in the affirmative.

Theorem 6.2. If graph G is $(\lfloor (p-2)/6 \rfloor) + 1$-extendable, then tough $(G) \geq 1$. ∎

It is interesting to compare Theorem 6.1 with a result due to Enomoto, Jackson, Katerinis and Saito [6]. Recall that a **k-factor** in a graph G is a spanning subgraph regular of degree k. Thus a perfect matching is just a 1-factor and a Hamiltonian cycle is a *connected* 2-factor.

Theorem 6.3. Let G be a graph with at least $n+1$ points and suppose tough $(G) \geq n$. Then, if $n|V(G)|$ is even, G has an n-factor. ∎

In order to compare Theorems 6.1 and 6.3, it is helpful to try to state them in as parallel a fashion as possible. With that in mind, consider the following two statements.

(A) tough $(G) \geq n \Rightarrow G$ has an n-factor.

(B) tough $(G) \geq n \Rightarrow G$ is $(n-1)$-extendable.

Note that if we enlarge our definition of n-extendability to say that a graph is **0-extendable** if it has a perfect matching, then when $n = 1$, both (A) and (B) say the same thing–namely, that G has a pm. (This, of course, is an immediate corollary of Tutte's 1-factor theorem [46].) For all $n \geq 2$, on the other hand, we claim that the two results are independent, in that neither implies the other. Graph families to establish this claim are constructed in [34].

7. Extendability in Planar Graphs

For this, the penultimate section of this paper, we return to Theorem 2.1 already presented in Section 2. This result represents the branch point for a second direction of research, the first being matching extension on surfaces of higher genus already discussed.

There are many 1-extendable *planar* graphs, for example, any cubic graph with no cutlines. (One does not even have to assume planarity for this result, essentially due to Petersen [32].) On the other hand, we know via Theorem 2.1 that no planar graph is 3-extendable. A second family of planar graphs which must be 1-extendable are those which are both 4-connected and even. (A graph is **even** if it has an even number of points.) This follows immediately from Tutte's deep theorem [47] which states that every 4-connected planar graph has a Hamilton cycle through any given line.

There remains the task of investigating the family of 2-extendable planar graphs.

An organized attack on this problem was first launched by Holton and the author [12]. First several constructions were presented which, when applied to two bicritical

(respectively, 2-extendable) graphs having points of degree 3, resulted in a larger bicritical (respectively, 2-extendable) graph having points of degree 3. It was at this point that a new graph parameter not previously studied in conjunction with bicriticality or n-extendability entered the picture–*cyclic connectivity*.

A graph G is **cyclically k-connected** if no set of $k-1$ or fewer lines, when deleted, disconnects the graph into 2 components each of which contains a cycle. In [12], the authors proved the following result about cubic 3-polytopes. (For more information on *polytopes*, the reader is directed to the classical book of Grünbaum [8]. Suffice it to say for our purposes that the 3-connected planar graphs are called *polytopal* because they are precisely the skeleta of 3-polytopes by a celebrated theorem of Steinitz [43].)

Theorem 7.1. If G is a cubic 3-connected planar graph which is cyclically-4-connected and has no faces of size 4, then G is 2-extendable. ∎

It should be mentioned that this result is sharp in the sense that there are cubic 3-polytopes which have no triangles or quadrilaterals, but which are only cyclically 3-connected and are not 2-extendable. On the other hand as well, there are cubic 3-polytopes which are cyclically 4-connected, but which are not 2-extendable. Of course by the preceding theorem, such graphs must contain a quadrilateral face. (For details, see [12].)

Let us also point out the fact that if a cubic cyclically 4-connected 3-polytope is also *bipartite*, then G can be shown to be 2-extendable. In other words, we can drop the restriction of no 4-cycles, if graph G is bipartite. This is a corollary of a stronger result on matching extension in (not necessarily planar) bipartite graphs due to Holton and the author [13].

The following is an immediate corollary of Theorem 7.1.

Corollary 7.2. If G is a cubic, 3-connected, cyclically 5-connected planar graph, then G is 2-extendable. ∎

Let us look next at graphs which are 5-connected and planar. The following result was proved independently by Lou [10] and by the author [37].

Theorem 7.3. If G is 5-connected planar and even, then G is 2-extendable. ∎

This result generalized an earlier result of Holton, Lou and the author [11] dealing with the 5-regular case. We state it as the following corollary.

Corollary 7.4. If G is 5-regular, 5-connected, even and planar, then G is 2-extendable. ∎

In view of Theorem 7.3, we now turn to the case when our planar (even) graph is 4-connected. The next result is an immediate corollary of Thomassen's generalization of the already mentioned Tutte theorem on 4-connected planar graphs. Thomassen [45] showed that every 4-connected planar graph is, in fact, Hamiltonian-connected. Our next result is an immediate corollary of Thomassen's theorem.

Theorem 7.5. If G is 4-connected, planar and even, then G is bicritical. ∎

So let us narrow down our question: Which 4-connected planar even graphs are 2-extendable?

Let us refer to the 5-point graph obtained by identifying exactly one point in each of two triangles as a **butterfly**. Let the point of identification (and hence of degree 4) be called the **body point**. It is trivial to see that if a graph G is 2-extendable (planar or not) then G can contain no point of degree 4 which serves as the body point of a butterfly subgraph of G. More generally, let $e_1 = u_1v_1$ and $e_2 = u_2v_2$ be two disjoint lines in a graph G such that $S = \{u_1, v_1, u_2, v_2\}$ is a point cutset of G. Suppose one of the components of $G - S$, call it C_1, is odd. Then let us call the subgraph of G induced by $S \cup V(C_1)$ a **generalized butterfly** (or **gbutterfly** in short). Clearly, a butterfly is just a gbutterfly in which the component C_1 contains precisely one point. Now it is also immediate that no 2-extendable graph can contain a gbutterfly. But it is not true that every 4-connected, planar, even graph G which contains no gbutterflies must be 2-extendable. A counterexample on 170 points is given in [37]. (It is not claimed that this counterexample is necessarily a smallest such. The 170-point graph described there was constructed so as to also satisfy the property of being 5-regular.)

So now let us return to the special case when the graphs in question are regular. Let us now once again include cyclic connectivity in our subsequent discussion.

Sachs [41] proved the following theorem about cyclic connectivity in 3-polytopal graphs.

Theorem 7.6. If G is 3-connected, planar and
(a) regular of degree 3, then $c\lambda(G) \leq 5$, while if
(b) G is regular of degree 4 or of degree 5, then $c\lambda(G) \leq 6$. ∎

On the other hand, Holton, Lou and the author [11] present a graph on 18 points which is 4-connected, 4-regular, even, gbutterfly-free and having cyclic connectivity $= 6$, but which is not 2-extendable. In other words, in view of Sachs' theorem, this counterexample has the highest possible cyclic connectivity.

Let us now try a different tack and consider *maximal* planar graphs (i.e., so-called *triangulations of the plane*).

First note that there are 3-connected maximal planar even graphs which do not even contain a perfect matching! One such is the *Kleetope* over the octahedron. (Given a plane graph G, one constructs the **Kleetope** over G by inserting a new point into the interior of each of the faces of G and then joining each new point to all points of said face of G.) On the other hand, from earlier in this section we know that 4-connected planar even graphs are 1-extendable and hence must have pm's. It turns out that in the 4-connected maximal planar even case, one need only forbid gbutterflies. In particular, we have the following [37].

Theorem 7.7. If G is a 4-connected maximal planar even graph containing no gbutterfly, then G is 2-extendable. ∎

It should be noted that there are examples of 4-connected maximal planar even graphs which contain gbutterflies and hence are not 2-extendable.

8. One More Result and Some Open Problems

Let us mention as our last result of this survey a very recent theorem due to Aldred, Holton and Lou [1]. We want to include it as it may be a first theorem in a new direction of studying how symmetries of graphs interact with matching extension.

Theorem 8.1. Let G be a cubic, cyclically k-connected for some $k \geq 3$ and line-transitive. Then either G is 2-extendable or it is the Petersen graph. ∎

We close with two interesting, but as yet unsolved, problems involving matching extension.

Let us define the **extendability** of a graph G to be the maximum n for which G is n-extendable.

Problem 1. Can the extendability of a graph be computed in polynomial time?

Our second and last problem deals with *maximal* n-extendable graphs.

Problem 2. Characterize those graphs G for which G is n-extendable, but $G \cup \{e\}$ is not, for all lines $e \in \overline{E(G)}$.

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